

A Note on Tachyons in the $D3 + \overline{D3}$ System

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Abstract

The periodic bounce of Born-Infeld theory of $D3$ -branes is derived, and the BPS limit of infinite period is discussed as an example of tachyon condensation. The explicit bounce solution to the Born-Infeld action is interpreted as an unstable fundamental string stretched between the brane and its antibrane.

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1 Introduction

The essential physics and basic equations of a system of $D3$ branes have been worked out in the paper of Callan and Maldacena [1]. The system of $D3$ branes is also one of the few systems permitting to a considerable extent explicit calculation of various aspects. Since, however, Ref. [1] appeared before Sen's work [2] focussed special interest on tachyons, the analysis of the $D3$ system in that direction is somewhat incomplete. In this note we comment on this, evaluate an elliptic integral and consider the limiting behaviour to the BPS state.

In most models constructed to explore the behaviour of tachyons in $Dp + \overline{Dp}$ systems, two initially unlinked branes are introduced with their own respective characteristics such as gauge fields and tachyons. The corresponding Lagrangian has generally not been written out and hence the $Dp + \overline{Dp}$ pair then has not been derived as a correspondingly explicit solution. This also does not permit a straightforward transition of the initially non-BPS branes to the BPS state. The $D3$ -brane theory of Ref. [1], however, allows this, at least to some extent, to be made very transparent. This is what we demonstrate in the following. The basic idea behind this is that the bounce, i.e. periodic solution of the equations of motion, has a period which in the limit of infinite size allows the solution to become a topological object like an instanton or domain wall configuration. This process can be described as tachyon condensation. The periodic solution thus interpolates between the branes of the brane pair. Thus one should find the periodic solution. In fact, a solution of this kind was given in Ref. [3] in the case of $D3$ without inclusion of the electric field, but the solution can be derived more generally. In the following we derive the full periodic solution and then demonstrate the transition to the BPS limit resulting in the fundamental string.

2 Bounces and the $D3 + \overline{D3}$ system

We recall briefly from Ref. [1] the points of relevance here. Taking the worldbrane gauge field as purely electric and considering the excitation of only one transverse coordinate which we here call y the worldbrane action reduces to

$$S = -\frac{1}{g_3} \int dt \int d^3x \sqrt{(1 - \mathbf{E}^2)(1 + \nabla y \cdot \nabla y) + (\mathbf{E} \cdot \nabla y)^2 - \dot{y}^2} \quad (1)$$

with $g_3 = g(2\pi)^3$, g the string coupling in the notation of Ref. [1]. In Ref. [1] the set of static solutions representing strings going between branes and antibranes is investigated by using spherical symmetry and choosing a single-charge electric solution of the associated

constraints. The equation for the static solution y can be integrated to give

$$y(r) = \int_r^\infty dr \frac{B}{\sqrt{r^4 - r_0^4}}, \quad (2)$$

where the length r_0 or throat radius (which originates as a constant of integration) is defined by

$$r_0^4 = B^2 - A^2. \quad (3)$$

The constant A is a parameter related to the electric point charge c_3 , from which the electric field originates and is related to the tension of the fundamental string, i.e.

$$c_3 \equiv g\pi = \pi(2\pi)^3 g_s \quad \text{and} \quad A = c_3, \quad (4)$$

where the constants and parameters are those used in Ref. [1] which we retain here for easy comparison. Thus with $A = 0$ we eliminate the electric field; the configuration described by the solution then remaining is that usually described as catenoid (see Ref. [3] and extensive references there).

Thus ignoring the electric field we have only the hull, i.e. catenoid part of the $D3$ -brane considered in detail in Ref. [3]. This is the solution

$$y_c(r) = \pm \int_r^\infty dr \frac{r_0^2}{\sqrt{r^4 - r_0^4}}. \quad (5)$$

With $x = r/r_0$, $\tilde{y}_c(x) = y_c(r_0 x)$, this integral can be rewritten and integrated as an elliptic integral so that

$$\begin{aligned} \tilde{y}_c(x) &= \pm \int_x^\infty \frac{dx}{\sqrt{x^4 - 1}} \\ &= \pm \frac{1}{\sqrt{2}} \left[K\left(\frac{\sqrt{2}}{2}\right) - \text{cn}^{-1}\left(\frac{1}{x}, \frac{\sqrt{2}}{2}\right) \right]. \end{aligned} \quad (6)$$

Here $\text{cn}(u, k)$ is the Jacobian elliptic cosine function (with the property that $\text{cn}(u, 0) = \cos u$) and $K(k)$ is its quarter period (with the property that $K(0) = \pi/2$). In these expressions the parameter k is the so-called elliptic modulus which determines the elliptic deviation of $2K(k)$ from π . The function $\text{cn}^{-1}(u, k)$ is the inverse function in analogy to its trigonometric counterpart. Properties of elliptic integrals and functions can be looked up in Refs. [4] and [5]. We set, k being the elliptic modulus,

$$a \equiv K(k), \quad k = \frac{\sqrt{2}}{2}.$$

When k has this value, one has (cf. [4], formula 111.10) $K = K' = K(k)$, $k' = \sqrt{1 - k^2}$. Then, since the variable or argument of the inverse function cn^{-1} is $1/x$, we can write the equation

$$x(y) = \left[\text{cn}\left(a \mp \sqrt{2}y, k\right) \right]^{-1}. \quad (7)$$

Suppressing the elliptic modulus k , we have

$$x(y) = \frac{1}{\text{cn}(K \mp \sqrt{2}y)}. \quad (8)$$

Now we use the formula (cf. Ref. [4], formula 122.03) $\text{cn}(u + K) = -k' \text{sdu}$, where $\text{sdu} = \text{sn}u/\text{dn}u$, the function $\text{dn}u$ being the third of the three fundamental Jacobian elliptic functions.

Hence

$$\frac{r(y)}{r_0} \equiv x(y) = \frac{1}{-k' \text{sd}(\mp \sqrt{2}y)} = \frac{\text{dn}(\mp \sqrt{2}y)}{-k' \text{sn}(\mp \sqrt{2}y)} = \frac{\text{dn}(\sqrt{2}y)}{\pm k' \text{sn}(\sqrt{2}y)}. \quad (9)$$

The function $\text{dn}(u)$ is a slowly varying function a little bit below 1 (see the figures in Ref. [5]; for $k^2 = 1/2$ the function $\text{dn}(u)$ varies between 1 and (approximately) 0.6 for any value of u). Thus roughly speaking it is almost a nonvanishing constant. We can concentrate on $\text{sn}(u)$. Depending on k this varies between a sine function and the hyperbolic tangent. Thus $x(y)$ has more or less the behaviour of $1/\sin(\sqrt{2}y)$. We show this schematically in Fig. 1.

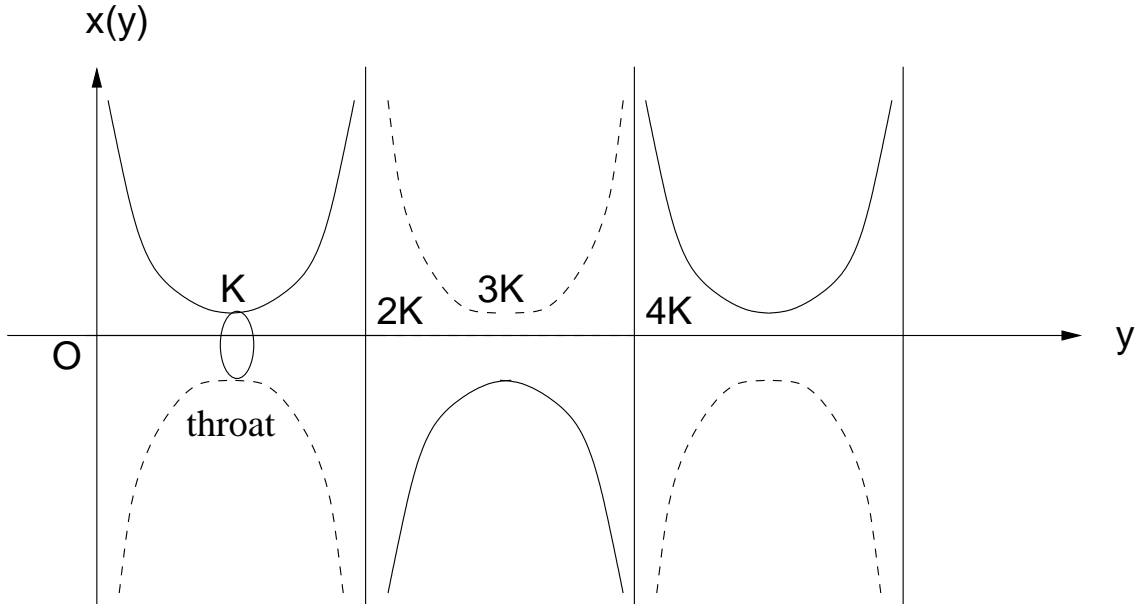


Fig. 1 Schematic behaviour of $x(y)$.

If we rotate the schematic sketch of $x(y)$ through 90 degrees, we obtain a figure like that of Fig. 2(c) of Ref. [1]. Thus one would like to know whether one can make the period so large that one brane is at infinity and we end up with the fundamental string. In the above the period is a fixed number which does not permit this. Comparing Eq. (5) with results of Ref. [1], however, we see that the full generalisation of our integral above for y_c , i.e. with inclusion of the electric field $E(r)$, is given by Eq.(2) above. Thus, in order to incorporate the electric field we have to make in the above the replacement

$$y_c(r) \longrightarrow \frac{X(r)r_0^2}{B}.$$

The product $g\pi$ is effectively the charge in the $D3$ brane from which the open string emanates (cf. Eq. (4) above). For $A = 0$ the action is a maximum, as pointed out in Ref. [1] and as demonstrated in Ref. [3]. The configuration y_c (i.e. that with $A = 0$) is therefore — in Ref. [1] — identified with that at the top of the potential (barrier). The classical configuration there is usually described as a sphaleron. Reference [1] therefore suggests that there must be configurations, i.e. solutions below this maximum, i.e. when $A \neq 0$, and these are the bounces or — as some say — periodic solitons. The authors of Ref. [1] do not evaluate any elliptic integrals. So if these periodic solutions exist, let's see them and obtain their period. Replacing y_c by the above expression in terms of X we have:

$$\frac{r(y)}{r_0} \longrightarrow \frac{r(X)}{r_0} = \frac{\text{dn}(\sqrt{2}Xr_0^2/\sqrt{r_0^4 + A^2})}{k'\text{sn}(\sqrt{2}Xr_0^2/\sqrt{r_0^4 + A^2})}. \quad (10)$$

We see that for $A = 0$ this reduces to the catenoid solution, i.e. to the configuration described as sphaleron in Ref. [1]. The solution $r(X)$ yields the periodic bounce. What is the period? From the literature (Refs. [4] and [5]) we take that

$$\text{sn}(u) \text{ has real period } 4K; \quad \text{dn}(u) \text{ has real period } 2K.$$

Here $\text{sn}(u, k)$ is the Jacobian elliptic sine function which together with the function $\text{cn}(u, k)$ satisfies the relation $\text{sn}^2 u + \text{cn}^2 u = 1$, which corresponds to $\sin^2 u + \cos^2 u = 1$. Thus $r(X)$ has period $4K$. This means, setting

$$u = \sqrt{2}Xr_0^2/\sqrt{r_0^4 + A^2},$$

that

$$\frac{\text{dn}(u + 4nK)}{\text{sn}(u + 4nK)} = \frac{\text{dn}(u)}{\text{sn}(u)},$$

and the period P in X is

$$P = \frac{4K\sqrt{r_0^4 + A^2}}{\sqrt{2}r_0^2} = \frac{4K}{\sqrt{2}}\sqrt{1 + \frac{A^2}{r_0^4}}. \quad (11)$$

Thus, replacing X by $X + P$, one has $\text{dn}(u) = \text{dn}(u + 4K)$.

Now we wish to calculate the smallest width of the throat, i.e. the smallest value of radius r indicated in Fig. 2.

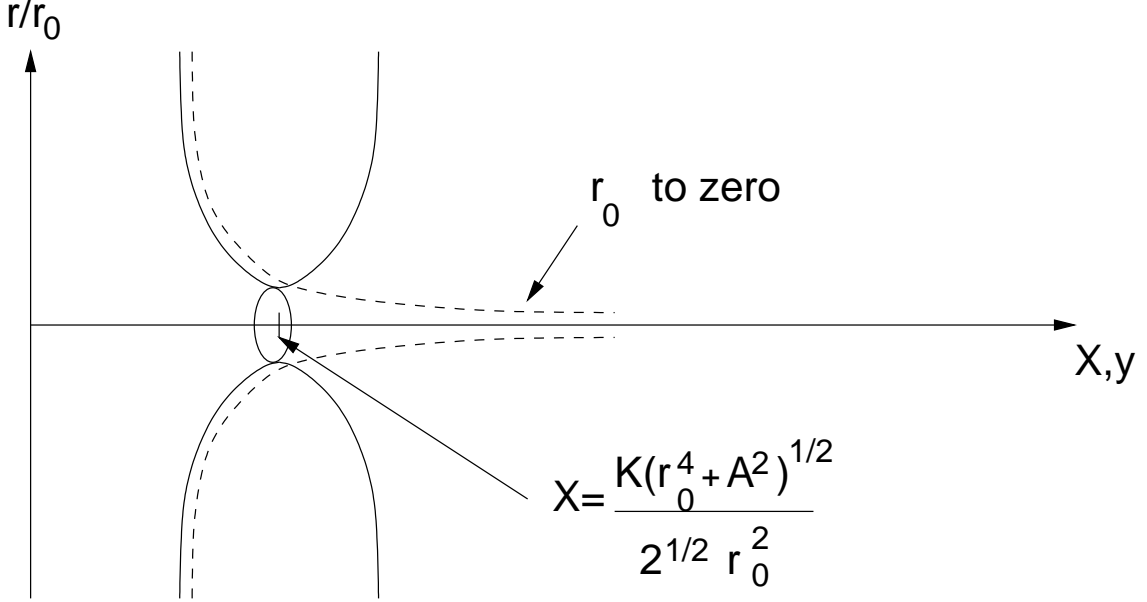


Fig. 2 The throat. Observe that for $r_0 \rightarrow 0$ the minimum wanders off to infinity but only for $A^2 \neq 0$.

From the tables of Ref. [5] we have:

$$\text{sn}(K) = 1, \quad \text{dn}(K) = k'.$$

Inserting this into $r(X)/r_0$, we obtain:

$$\frac{r(K)}{r_0} = 1$$

with

$$\frac{\sqrt{2} X r_0^2}{\sqrt{r_0^4 + A^2}} = K, \quad \frac{K \sqrt{r_0^4 + A^2}}{\sqrt{2} r_0^2} = \frac{P}{4}. \quad (12)$$

We observe that for $r_0 \rightarrow 0$: $X \rightarrow \infty$ and the antibrane goes to infinity, at the end of the fundamental string. This is the passage to the supersymmetric or BPS limit, as discussed in Refs. [1] and [3]. Thus we interpret the spike carrying the electric field (originating from the charge at $r = 0$) as the open string emanating from one brane and heading towards

the other. This suggests to identify the fluctuations about the brane-antibrane pair, i.e. about the periodic solution, with the fluctuations of the open string, after all, it is the low energy fluctuations which replace the open string in our considerations here. Hence it is not only the negative-eigenvalue or ground-state mode which has to be identified with the tachyon, but correspondingly also the zero-eigenvalue or first-excited-state mode which has to be identified with (here) the electric field.

The next step would be to look at the equation of small fluctuations about the periodic solution above. Then, by equating to zero the electric field or charge, one should arrive at the fluctuation equation about the catenoid — this is the equation with differential operator given by Eq. (33) in Ref.[3] — and by imposing the BPS conditions (Eq. (64) of Ref.[3]) one should obtain the fluctuation equation about the BPS solution, i.e. Eq. (74) of Ref.[3]. The procedure to derive this equation would be that of Ref.[3] by following the steps after Eq. (62), however without imposing immediately the BPS conditions there given by Eq. (64). This seems to require some longer but straightforward calculations. We skip these here and instead — after the following discussion — present plausibility arguments based on analogous but easily solvable examples.

Thus we arrive at the following picture. The brane-antibrane pair with the brane and antibrane a finite distance apart is a periodic solution of the Euler-Lagrange equations obtained from the Born-Infeld Lagrangian. Since in the two limiting cases of BIon (no field y) and catenoid (no gauge field) the solution is either stable or unstable as discussed in detail in Ref. [3], in the general case the brane-antibrane pair is unstable and hence the equation of small fluctuations about this solution must possess a negative eigenvalue, as is wellknown (recall that a periodic solution has the shape of a squeezed instanton–antiinstanton pair which pushes one of these to infinity when the period is made infinite, thereby removing the negative mode). Arguments of translation invariance imply that there must also be a zero mode, i.e. a solution of the small fluctuation equation with eigenvalue zero. These solutions describe the amplitudes of the fluctuations in the direction perpendicular to the $D3$ branes. We recall that at low energies the effect of an open string, i.e. its fluctuations, is approximated by its lowest eigenmodes, and these are the tachyon and gauge field modes with negative mass-squared and zero mass respectively. Thus it is suggestive to identify these lowest oscillations of the open string with the lowest fluctuations about the brane-antibrane pair in the above sense. A scalar tachyon field ϕ was not inserted into the theory originally, however, the tachyon mode arising here can be considered as a remnant of such a field as a function of one co-dimension orthogonal to the brane. A similar though different consideration may apply to the gauge field and the zero mode. These aspects have been discussed in Ref. [6].

Since we do not enter here into a detailed study of the small fluctuation equation and its spectrum, we restrict ourselves — as stated above — to arguments of analogy. The simplest example of a nonperiodic bounce is obtained for the (admittedly unphysical) cubic potential as demonstrated in Ref. [7] and the fluctuation equation is the equation with the well-known Pöschl–Teller potential (of $1/\cosh^2 x$ type with $x \in \mathbf{R}$). The solution (i.e. “zero mode”) associated with the eigenvalue zero is, as usual for reasons of translational invariance, given by the derivative of the classical configuration and when this is an odd function the ground state has a negative eigenvalue. These general properties of bounces have been discussed by Coleman [8]. In fact Coleman has given general arguments to the effect that a bounce is associated with one physical negative eigenvalue of the fluctuation equation.

Periodic configurations arise in periodic cases. Thus in Ref. [9] the very popular examples of double-well potential, inverted double-well potential and cosine potential were considered on a circle and the nontopological, periodic instanton-like configurations were derived. Very naturally these solutions were found to be Jacobian elliptic functions like the function $\text{sn}(u, k)$ in the case of the double well potential. Such a solution reduces in the limit of $k = 1$ to the well-known topological kink solution, i.e. $\tanh x$, and therefore is not a topological vacuum for other values of k . The effect on the quarter period $K(k)$ of allowing k to vary from 0 to 1 is to vary from $\pi/2$ to $K(1) = \infty$. Thus the periodic solution varies from a trigonometric form to the hyperbolic tangent. Considering now fluctuations about these configurations, one obtains — as explicitly demonstrated in Ref. [8] — as fluctuation equation the wellknown Lamé equation whose solutions can be looked up in books like Ref. [10]. In each case these equations have several solutions with negative, zero and positive eigenvalues, some of which merge together in the limit $k \rightarrow 1$ with the disappearance of negative modes in topological cases.

3 Concluding remarks

As noted, in the $D3 + \overline{D3}$ system discussed above the tachyon field and tachyon potential do not appear explicitly, instead only indirectly through the negative mode. Models dealing with explicit tachyon potentials, such as the open string field theory models of Ref. [6], permit the explicit derivation of the fluctuation modes. In these cases the tachyon potential is either of the unbounded cubic type (in the bosonic theory) or of double-well type (in the superstring theory). The fluctuation spectra of these potentials have been considered in detail in Refs. [7] and [9] respectively.

We know from other studies in Ref. [9] that the negative eigenvalue mode merges

with the zero mode in the limit in which the period of the periodic solution or bounce becomes infinite, i.e. when the periodic solution becomes a soliton. In effective tachyon scalar superstring theories as discussed for instance in Ref. [6] the calculations are essentially those familiar from soliton theory, the tachyon scalar field depending on only one co-dimension which is orthogonal to the remaining spatial coordinates of the original Dp -brane. With the tunneling through the central hump of the double-well potential, the original perturbation vacuum energy is lowered to the real ground state, which is characterised by one of the two topological and hence BPS states provided by the kink or domain wall solution. The final state thus reached by “tachyon condensation” then is the $D(p - 1)$ -brane which carries a topological charge. In this process the period of the periodic solutions becomes infinite. This seems to be the process also described in Ref. [11], where periodicity arises by putting the domain wall on a circle as demonstrated explicitly in several cases in Ref. [9]. A situation analogous to the $D3$ case considered here can be seen in the case of $D2$ -branes, although there the reverse process has been discussed, i.e. that of strings tunneling to branes, as in Refs. [12] and [13].

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References

- [1] C.G. Callan and J. M. Maldacena, *Brane dynamics from the Born-Infeld action*, Nucl. Phys. **B513** (1998) 198 [arXiv: hep-th/9708147].
- [2] A. Sen, *Stable non-BPS states in string theory*, JHEP **08** (1998) 007 [arXiv: hep-th/9803194].
- [3] D.K. Park, S. Tamaryan, H.J.W. Müller-Kirsten and Jian-zu Zhang, *D-branes and their absorptivity in Born-Infeld theory*, Nucl. Phys. **B594** (2001) 243 [arXiv: hep-th/0005165].
- [4] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Second Ed. (Springer, 1971), p.252.
- [5] L.M. Milne-Thomson, *Jacobian Elliptic Function Tables*, (Dover Publ., 1950).
- [6] K. Ohmori, *A review on tachyon condensation in open string field theories*, arXiv: hep-th/0102085. See sect. 4.1.
- [7] H.J.W. Müller-Kirsten, Jian-zu Zhang and D.H. Tchrakian, *Stability or metastability and eigenvalues of the equation of small fluctuations*, Int. J. Mod. Phys. **A5** (1990) 1319.
- [8] S. Coleman, in: *The Whys of Subnuclear Physics*, ed. A. Zichichi (Plenum, 1979), p. 805.
- [9] J.-Q. Liang, H.J.W. Müller-Kirsten and D.H. Tchrakian, *Solitons, bounces and sphalerons on a circle*, Phys. Lett. **B 282** (1992) 105.
- [10] F.M. Arscott, *Periodic Differential Equations* (Pergamon, 1964).
- [11] J. Schwarz, *TASI Lectures on Non-BPS D-Brane Systems*, arXiv: hep-th/9908144. See sect. 3.3.
- [12] R. Emparan, *Born-Infeld strings tunneling to D2-branes*, Phys. Lett. **B 423** (1998) 71 [arXiv: hep-th/9711106].
- [13] D.K. Park, S. Tamaryan and H.J.W. Müller-Kirsten, *D2-branes with magnetic flux in the presence of RR fields*, Nucl. Phys. **B635** (2002) 192 [arXiv: hep-th/0111026].